RATIONALITY CRITERIA FOR MOTIVIC ZETA FUNCTIONS

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1. Introduction

Work over $\mathbb{C}$. Consider $K_0\mathrm{Var}_{\mathbb{C}}$, the Grothendieck ring of varieties. As an abelian group, $K_0\mathrm{Var}_{\mathbb{C}}$ is generated by isomorphism classes of varieties, modulo the relation $[X] = [Y] + [U]$ where Y is a closed subvariety of X and $U = X - Y$. The multiplicative structure is induced by multiplication of varieties. A **motivic measure** is a ring homomorphism $K_0\mathrm{Var}_{\mathbb{C}} \to A$ for some ring A.

For X a variety, denote $\mathrm{Sym}^n(X)$ to be the n-th symmetric product of X. Define X's **motivic zeta function** to be

$$\zeta_X(t) = \sum_{n\geq 0}[\mathrm{Sym}^n(X)]t^n \in 1 + tK_0\mathrm{Var}_{\mathbb{C}}[[t]].$$

The motivic zeta function was first defined by Kapranov [Kap00]. In that paper, Kapranov showed that $\zeta_X(t)$ is rational when X is a curve and asked whether rationality holds in general. Larsen and Lunts [LL03] negatively answered Kapranov's question and proved that $\zeta_X(t)$ is not rational when X is a complex surface with geometric genus ≥ 2. In [LL04], Larsen and Lunts strengthened their result and showed that when X is a complex surface, $\zeta_X(t)$ is rational iff $\kappa(X) = -\infty$.

This expository paper reviews the results of [LL04].

2. Rationality of Power Series

Before proving rationality or irrationality results, we need to define what rationality of power series means. This is actually non-trivial, as we will see in this section.

Let R be a commutative ring and $f \in R[[t]]$. There are several different notions of rationality of f.

Definition 2.1. f is called **globally rational** if there exists $g, h \in A[t]$ such that f is the unique solution to the equation $gx = h$ in $A[[t]]$.

Definition 2.2. f is called **determinantally rational** if there exists $n \in \mathbb{N}$ such that

$$\det \begin{bmatrix} a_i & a_{i+1} & \cdots & a_{i+m} \\ a_{i+1} & a_{i+1} & \cdots & a_{i+m+1} \\ \cdots & \cdots & \ddots & \cdots \\ a_{i+m} & a_{i+m+1} & \cdots & a_{i+2m} \end{bmatrix} = 0$$

for large enough i.

Definition 2.3. f is called **pointwise rational** if for any ring homomotphism $\phi : A \to K$ where K is a field, $\phi(f) \in K[[t]]$ is pointwise rational.

Proposition 2.4. *Globally rational implies determinantally rational. Determinantally rational implies pointwise rational. When R is a domain, the three definitions of rationality agree. In general, pointwise rational does not imply determinantally rational and determinantally rational does not imply globally rational.*

It has been shown by Poonen that $K_0\mathrm{Var}_{\mathbb{C}}$ is not a domain ([Poo02]). So it is not meaningless to distinguish the three different notions of rationality. All the results of Larsen and Lunts are in the strongest sense: the rationality results say that the motivic zeta functions are globally rational, and the irrationality results say that the motivic zeta functions are not pointwise rational.

3. Rationality Result

In this section we prove the main rationality result in [LL04].

Theorem 3.1 ([LL04], Theorem 3.9)**.** *$\zeta_X(t)$ is globally rational when X is a complex surface with $\kappa(X) = -\infty$.*

We go over the proof.

Proposition 3.2. *Let X be a variety, $Y \subseteq X$ be a closed subvariety. $U = X - Y$. Then*

$$[\mathrm{Sym}^n(X)] = \sum_{0\le i\le n} [\mathrm{Sym}^i(Y)][\mathrm{Sym}^{n-i}(U)].$$

Corollary 3.3. *In the setting of Proposition 3.2, we have $\zeta_X(t) = \zeta_Y(t)\zeta_U(t)$. So if two of $\zeta_X(t)$, $\zeta_Y(t)$, $\zeta_U(t)$ are globally (resp. pointwise) rational, then the third is also globally (resp. pointwise) rational.*

We need some results about motivic zeta functiosn of vector bundles.

Lemma 3.4. *Let X be a variety and $E \to X$ be a Zariski-locally trvial fiber bundle with fiber F. Then $[E] = [X][F]$.*

The following proposition is by Totaro [Göt03].

Proposition 3.5 ([Göt03], Lemma 4.4)**.** *Let X be a variety and E be a vector bundle over x with rank r. Then $[\mathrm{Sym}^n E] = [\mathrm{Sym}^n X]\mathbb{L}^{rn}$.*

Proof Sketch. First observe that we can assume E is a trivial vector bundle. Then by trivial induction we can assume $r = 1$. The main part of the proof is stratifying $\mathrm{Sym}^n X$ according to the partition of n corresponding to each n-tuple in $\mathrm{Sym}^n X$ and proving the result on each strata. □

Totaro's result together with Lemma 3.4 immediately implies the following corollary.

Corollary 3.6. *In the setting of Proposition 3.5, we have $\zeta_E(t) = \zeta_X(\mathbb{L}^r t)$. In particular, if $\zeta_X(t)$ is globally (resp. pointwise) rational, then $\zeta_E(t)$ is globally (resp. pointwise) rational.*

By observing that $[\mathbb{P}^r] = 1 + \mathbb{L} + \cdots + \mathbb{L}^r$, we can prove the following result.

Corollary 3.7. *Let X be a variety and $P \to X$ be a Zariski-locally trivial projective bundle of rank r. Then $\zeta_P(t) = \zeta_X(t)\zeta_X(\mathbb{L}t)\cdots\zeta_X(\mathbb{L}^r t)$. In particular, if $\zeta_X(t)$ is globally (resp. pointwise) rational, then $\zeta_P(t)$ is globally (resp. pointwise) rational.*

Kapranov [Kap00] proved that the motivic zeta functions for curves are rational in $1 + t\mathcal{M}_{\mathbb{C}}[[t]]$, where $\mathcal{M}_{\mathbb{C}} = (K_0\mathrm{Var}_{\mathbb{C}})_{\mathbb{L}}$. The invertibility of $\mathbb{L}$ is needed because Kapranov's proof is based on motivic integration. However, the proof can be easily modified into a proof for $K_0\mathrm{Var}_{\mathbb{C}}$.

Theorem 3.8 (Kapranov)**.** *$\zeta_X(t)$ is globally rational when X is a curve.*

Proof Sketch. By Corollary 3.3, we can assume X is smooth projective. For $n \geq 2g-1$, we have a map $\mathrm{Sym}^n X \to \mathrm{Jac}^0 X$ which realizes $\mathrm{Sym}^n X$ as a projective bundle over $\mathrm{Jac}^0 X$. We also have maps between projective bundles $\mathrm{Sym}^{n-1} X \to \mathrm{Sym}^n X$. The complement of the image is a vector bundle over $\mathrm{Jac}^0 X$. So we have

$$[\mathrm{Sym}^{n+1} X] - [\mathrm{Sym}^n X] = [\mathrm{Jac}^0 X]\mathbb{L}^{n+1-g}.$$

Trivial calculation shows that $\zeta_X(t)(1-t)(1-\mathbb{L}t)$ is a polynomial of degree $\leq 2g$. □

By Kapranov's theorem and Corollary 3.3, we have

Corollary 3.9. *The rationality of $\zeta_X(t)$ when X is a surface depends only on the birational class of X.*

Now we can easily prove the main rationality result.

Proof of Theroem 3.1. We have birational classification of complex surfaces. When $\kappa(X) = -\infty$, we know that X is birationally equivalent to $\mathbb{P}^1 \times C$ where C is a curve. The rationality of $\mathbb{P}^1 \times C$ follows from Theorem 3.8 and Corollary 3.7. □

4. Preparations for the Irrationality Result

The remaining of this expository paper is devoted to the proof of the main irrationality result in [LL04].

Theorem 4.1 ([LL04], Theorem 7.6)**.** *A complex surface X with $\kappa(X) \geq 0$ has $\zeta_X(t)$ not pointwise rational.*

The proof is by constructing a motivic measure $\mu : K_0\mathrm{Var}_{\mathbb{C}} \to R$ (where R is a domain) that factors through $\mathbb{Z}[\mathrm{SB}]$, and then proving that $\mu(\zeta_X(t)) \in 1 + tR[[t]]$ is not rational. To define the motivic measure, we need the theory of λ-rings.

Definition 4.2. A **λ-ring** is a commutative ring R equipped with a sequence $\lambda^0, \lambda^1, \ldots$ of set-functions $R \to R$, such that

(1) $\lambda^0(x) = 1$;
(2) $\lambda^1(x) = x$;
(3) $\lambda^n(x+y) = \sum_{0\leq i\leq n} \lambda^i(x)\lambda^{n-i}(y)$.

Definition 4.3. A **special λ-ring** is a λ-ring R such that

(1) $\lambda^n(xy) = P_n(\lambda^1 x, \ldots, \lambda^n x, \lambda^1 y, \ldots, \lambda^n y)$.
(2) $\lambda^m \lambda^n(x) = P_{m,n}(\lambda^1 x, \ldots, \lambda^{mn} x)$.

In the definition, P_n and $P_{m,n}$ are some universal polynomials with coefficients in $\mathbb{Z}$.

Remark 4.4. In some literature, λ-rings are called "pre-λ-rings" and special λ-rings are called "λ-rings".

Definition 4.5. Let R be a λ-ring. We define the **Adams operations** $\psi^n : R \to R$ as

$$\psi^n(x) = (-1)^{n+1} \sum_{0 \le i \le n} i\lambda^i(x)\lambda^{n-i}(-x).$$

Proposition 4.6. *Several properties of* ψ^n.

1. ψ^n *is a polynomial in* λ^i, $0 \le i \le n$.
2. ψ^n *is a ring homomorphism when* R *is special.*
3. $\psi^n(x) = x^n$ *when* x *is a one-dimensional element, i.e.* $\lambda^i(x) = 0$ *for* $i \ge 2$.

Now we define the λ-ring that is used in constructing the motivic measure.

Definition 4.7. Let X be a variety. Define $\overline{K}(X)$ to be the abelian group generated by classes of vector bundles on X, modulo the relation $[M] = [N] + [P]$ when $M \simeq N \oplus P$. Multiplication on $\overline{K}(X)$ is multiplication of vector bundles. Lambda operations on $\overline{K}(X)$ are exterior powers of vector bundles.

Remark 4.8. The usual $K(X)$ is a quotient of $\overline{K}(X)$ as λ-rings.

We need $\overline{K}(X)$ instead of $K(X)$ because we have a group homomorphism $\overline{K}(X) \to \mathbb{Z}$ by taking the dimension of the global sections.

It is well-known that $K(X)$ is a special λ-ring by using the splitting principle. However, the splitting principle only produces short exact sequences, which in general do not split. Larsen and Lunts proved that $\overline{K}(X)$ is special in a different way.

Theorem 4.9 ([LL04], Theorem 5.1)**.** $\overline{K}(X)$ *is special.*

Proof Sketch. Note that the conditions in Definition 4.3 only involve two elements x and y. For arbitrary x and y, we construct a homomorphism from some special λ-ring to $\overline{K}(X)$, whose image contains x and y. Then we know that the conditions are satisfied.

The special λ-ring is chosen to be R^2, the free special λ-ring with two generators, which can be characterized using representation rings of the symmetric groups. An explicit homomorphism $R^2 \to \overline{X}$ that sends the generators to x and y is not difficult to construct. ☐

The main result of [LL03], which is a characterization of $K_0\mathrm{Var}_{\mathbb{C}}/\mathbb{L}$, is needed in the proof of the irrationality result.

Definition 4.10. For two varieties X, Y, say X and Y are **stably birational** if $X \times \mathbb{P}^k$ is birational to $Y \times \mathbb{P}^l$ for some k, l. Define SB to be the set of stable birational classes in $\mathrm{Var}_{\mathbb{C}}$. SB equipped with multiplication of varieties is a commutative monoid.

Theorem 4.11 ([LL03], Theorem 2.3, Proposition 2.8)**.** *There is a ring homomorphism* $K_0\mathrm{Var}_{\mathbb{C}} \to \mathbb{Z}[\mathrm{SB}]$ *which sends the class of a variety to its stable birational class, and the kernel is* $\langle \mathbb{L} \rangle$.

5. Irrationality Result

In this final section we prove the irrationality result.

The first step of the proof is to construct a sequence of motivic measures.

Definition 5.1. Let $M = 1+s\mathbb{Z}[s]$ be the commutative monoid of polynomials with coefficients in $\mathbb{Z}$ and constant 1, equipped with multiplication of polynomials. Let $\mathbb{Z}[M]$ be the monoid ring. For $n \geq 1$, define motivic measure $\mu_n : K_0\mathrm{Var}_{\mathbb{C}} \to \mathbb{Z}[M]$ by

$$\mu_n(X) = \sum_{0\leq i\leq \dim X} h^0(X, \psi^n\Omega^i_X)s^i.$$

Proposition 5.2. *Properties of* μ_n.

1. μ_n *is birational invariant.*
2. $\mu_n(X \times Y) = \mu_n(X)\mu_n(Y)$.
3. $\mu_n(\mathbb{P}^k) = 1$.

Combining the proposition with Theorem 4.11, we get

Corollary 5.3. μ_n *factors through* $\mathbb{Z}[\mathrm{SB}]$.

We would like to prove that for some n, $\mu_n(\zeta_X(t))$ is irrational. In the formula, we have terms involving $\Omega_{\mathrm{Sym}^m X}$. $\mathrm{Sym}^m X$ is not smooth in general, so we would like a smooth replacement of it. It is known that $\mathrm{Hilb}^m X$ is smooth when X is a smooth surface, and that $\mathrm{Hilb}^m X$ and $\mathrm{Sym}^m X$ are closely related. The following theorem of Göttsche makes the replacement possible.

Theorem 5.4 ([Göt03], Theorem 1.1)**.**

$$[\mathrm{Hilb}^n X] = \sum_{\alpha\in P(n)} [\mathrm{Sym}^\alpha X]\mathbb{L}^{n-|a|}.$$

In the formula, $P(n)$ *is the set of partitions of* n*. Each* $\alpha \in P(n)$ *is written as* $(1^{\alpha_1} \cdots n^{\alpha_n})$*.* $|a| = \sum_i \alpha_i$ *and* $\mathrm{Sym}^\alpha X = \prod_i \mathrm{Sym}^{\alpha_i} X$*.*

Corollary 5.5. *In* $\mathbb{Z}[\mathrm{SB}]$*,* $[\mathrm{Hilb}^n X] = [\mathrm{Sym}^n X]$*.*

We need three more propositions.

Proposition 5.6 ([LL04], Proposition 7.2, Proposition 7.3)**.**

$$H^0(\mathrm{Hilb}^m X, \omega^{\otimes n}_{\mathrm{Hilb}^m X}) = \mathrm{Sym}^m H^0(X, \omega^{\otimes n}_X).$$

Proof Sketch. It is easy to show that

$$H^0(X^m, \omega^{\otimes n}_{X^m})^{S_m} = \mathrm{Sym}^m H^0(X, \omega^{\otimes n}_X).$$

So we only need to prove

$$H^0(\mathrm{Hilb}^m X, \omega^{\otimes n}_{\mathrm{Hilb}^m X}) = H^0(X^m, \omega^{\otimes n}_{X^m})^{S_m}.$$

This is by

1. restricting to an open subset of Hilb^m whose complement is of codimension 2,
2. injecting the sheaves on two sides into a larger sheaf on some variety,
3. proving that the images coincide.

□

Proposition 5.7 ([LL04], Proposition 7.1, Proposition 7.5)**.** *The coefficients of* $\mu_n(\mathrm{Hilb}^m X)$ *are bounded independent of* m*.*

Proof Sketch. $\psi^n\lambda^i$ are polynomials in λ^j, and exterior powers are summands of tensor products. So we only need to prove that $h^0(\mathrm{Hilb}^m X, (\Omega^1_{\mathrm{Hilb}^m X})^{\otimes n})$ are bounded independent of m. It is easy to prove that

$$H^0(\mathrm{Hilb}^m X, (\Omega^1_{\mathrm{Hilb}^m X})^{\otimes n}) \subseteq H^0(X^m, (\Omega^1_{X^m})^{\otimes n})^{S_m}.$$

So we only need to bound the right hand side.

$(\Omega^1_{X^m})^{\otimes n}$ can be decomposed as a direct sum of tensor products of pullbacks of Ω^1_X from different factors of X^m. The direct sum is over $\{1, \dots, n\}^m$, and by considering S_m action we can simplify it into a direct sum over partitions of n. Finally we see that in each summand of the direct sum, H^0 do not depend on m. □

Proposition 5.8 ([LL04], Theorem 2.9)**.** *Let G be a free abelian group and $F = \mathrm{Frac}\ \mathbb{Z}[G]$. Then a power series $f = \sum_{i\geq 0} g_i t^i \in G[[t]] \subseteq F[[t]]$ is rational iff there exists $n \geq 1$ and $h_0, \dots, h_{n-1} \in G$ such that for $g_{i+n} = h_{i \bmod n} g_i$ for i large enough.*

Proof of Theorem 4.1. $\kappa(X) \geq 0$, so we can choose n such that $H^0(X, \omega_X^{\otimes n}) \neq 0$. By Proposition 5.6, $H^0(\mathrm{Hilb}^m X, \omega^{\otimes n}_{\mathrm{Hilb}^m X}) \neq 0$. In particular, this implies that the degree of $\mu_n(\mathrm{Hilb}^m X)$ is $2m$.

Assume that $\mu_n(\zeta_X(t))$ is rational. Let G be the group completion of M. Then by Proposition 5.8, there exists some p and $h_0, \dots, h_{p-1} \in G$, such that $\mu_n(\mathrm{Hilb}^{m+p} X) = h_{m \bmod p} \mu_n(\mathrm{Hilb}^m X)$ for m large enough. $\mu_n(\mathrm{Hilb}^m X) \in M$ for all m, so $h_i \in M$. $\mu_n(\mathrm{Hilb}^m X)$ has degree $2m$ and constant term 1, so h_i is not a monomial.

For m large enough, and all $k \geq 0$, we have $\mu_n(\mathrm{Hilb}^{m+kp} X) = h^k_{m \bmod p} \mu_n(\mathrm{Hilb}^m X)$. h_i's are not monomials, so the coefficients of $\mu_n(\mathrm{Hilb}^{m+kp} X)$ are unbounded as k goes to ∞. This contradicts with Proposition 5.7. □

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